

# Application of a “Jacobi identity” for vertex operator algebras to zeta values and differential operators

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*Dedicated to the memory of Moshé Flato*

## Abstract

We explain how to use a certain new “Jacobi identity” for vertex operator algebras, announced in a previous paper (math.QA/9909178), to interpret and generalize recent work of S. Bloch’s relating values of the Riemann zeta function at negative integers with a certain Lie algebra of operators.

## 1 Introduction

In [L2] we have announced some results relating vertex operator algebra theory to certain phenomena associated with values of the Riemann zeta function  $\zeta(s)$  at negative integers, and we also presented expository background. In the present paper, a continuation, we announce and sketch the proofs of more of these results. The goal is to use vertex operator algebra theory to interpret the classical formal equality

$$\sum_{n>0} n^s = \zeta(-s), \quad s = 0, 1, 2, \dots, \quad (1.1)$$

where the left-hand side is the divergent sum and the right-hand side is the zeta function (analytically continued), and to exploit the interpretation in the setting of vertex operator algebra theory. More specifically, we wanted to “explain” some recent work [Bl] of S. Bloch’s involving zeta values and

central extensions of Lie algebras of differential operators. Below we shall summarize the contents of [L2] and make some additional comments, and then in Section 3 we shall present the new results. These deal with the passage from the general “Jacobi identity” announced in [L2], Theorem 4.2 (recalled in Theorem 2.2 below), to the more special formula, Theorem 3.1 of [L2] (recalled in Theorem 1.1 below). This in turn recovered the main formulas in [Bl], expressing the structure of a Lie algebra, constructed using zeta-function values from certain vertex operators, providing a central extension of a Lie algebra of differential operators. The work [L3] contains details and related results.

While it might help the reader of the present paper to have the paper [L2] available to refer to, we shall review some background and motivation in this extended Introduction and in Section 2 below, to make this paper more self-contained. The famous classical “formula”

$$1 + 2 + 3 + \cdots = -\frac{1}{12}, \quad (1.2)$$

which has the rigorous meaning

$$\zeta(-1) = -\frac{1}{12}, \quad (1.3)$$

is intimately related to the regularizing of certain infinities in conformal field theory. In [L2] we announced some general principles of vertex operator algebra theory that elucidate the passage from the unrigorous but suggestive formula (1.2) to formula (1.3), and the generalization (1.1), in the process “explaining” some results in [Bl]. Foundational notions of vertex operator algebra theory, and “formal calculus,” as presented in [FLM] and [FHL], enter in an essential way.

Our main themes involve: the ubiquity of generating functions—the introduction of new formal variables and generating functions in order to render complicated things easier, more natural, and at the same time, much more general, as in the corresponding parts of [FLM]; the use of commuting formal variables rather than complex variables because they provide the most natural way to handle the doubly-infinite series such as  $\delta(x) = \sum_{n \in \mathbb{Z}} x^n$  that pervade the natural formulations and proofs in the subject; the exploitation of the formal exponential of the differential operator  $x \frac{d}{dx}$  as a formal change-of-variables automorphism (again as in [FLM]); the formulation of Euler’s

interpretation of the divergent series (1.1) by means of the operator product expansion in conformal field theory; and the consideration of central extensions of Lie algebras of differential operators in terms of the very general context of what we termed the “Jacobi identity” [FLM] for vertex operator algebras.

Now we sketch the relevant classical background on the Virasoro algebra and related matters. Consider the Lie algebra

$$\mathfrak{d} = \text{Der } \mathbb{C}[t, t^{-1}] \quad (1.4)$$

of formal vector fields on the circle, with basis  $\{t^n D | n \in \mathbb{Z}\}$ , where

$$D = D_t = t \frac{d}{dt}, \quad (1.5)$$

and recall the Virasoro algebra  $\mathfrak{v}$ , the central extension

$$0 \rightarrow \mathbb{C}c \rightarrow \mathfrak{v} \rightarrow \mathfrak{d} \rightarrow 0, \quad (1.6)$$

where  $\mathfrak{v}$  has basis  $\{L(n) | n \in \mathbb{Z}\}$  together with a central element  $c$ ; the bracket relations are given by

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{m+n,0}c \quad (1.7)$$

and  $L(n)$  maps to  $-t^n D$  in (1.6). This Lie algebra is naturally  $\mathbb{Z}$ -graded, with  $\deg L(n) = n$  and  $\deg c = 0$ . It has the following well-known realization: Start with the Heisenberg Lie algebra with basis consisting of the symbols  $h(n)$  for  $n \in \mathbb{Z}$ ,  $n \neq 0$  and a central element 1, with the bracket relations

$$[h(m), h(n)] = m\delta_{m+n,0}1. \quad (1.8)$$

For convenience we adjoin an additional central basis element  $h(0)$ , so that the relations (1.8) hold for all  $m, n \in \mathbb{Z}$ . This Lie algebra acts irreducibly on the polynomial algebra

$$S = \mathbb{C}[h(-1), h(-2), h(-3), \dots] \quad (1.9)$$

as follows: For  $n < 0$ ,  $h(n)$  acts by multiplication; for  $n > 0$ ,  $h(n)$  acts as  $n \frac{\partial}{\partial h(-n)}$ ;  $h(0)$  acts as 0; and 1 acts as the identity operator. Then  $\mathfrak{v}$  acts on  $S$  via:

$$c \mapsto 1, \quad (1.10)$$

$$L(n) \mapsto \frac{1}{2} \sum_{j \in \mathbb{Z}} h(j)h(n-j) \quad \text{for } n \neq 0, \quad (1.11)$$

$$L(0) \mapsto \frac{1}{2} \sum_{j \in \mathbb{Z}} h(-|j|)h(|j|). \quad (1.12)$$

In the case of  $L(0)$ , the absolute values make the operator well defined, while for  $n \neq 0$  the operator is well defined as it stands, since  $[h(j), h(n-j)] = 0$ ; thus the operators (1.11) and (1.12) are in “normal-ordered form,” that is, the  $h(n)$  for  $n > 0$  act to the right of the  $h(n)$  for  $n < 0$ . Using colons to denote normal ordering, we thus have

$$L(n) \mapsto \frac{1}{2} \colon \sum_{j \in \mathbb{Z}} h(j)h(n-j) \colon \quad (1.13)$$

for all  $n \in \mathbb{Z}$ . It is well known that the operators (1.13) indeed satisfy the bracket relations (1.7). (This exercise and the related constructions are presented in [FLM], for example, where the standard generalization of this construction of  $\mathfrak{v}$  using a Heisenberg algebra based on a finite-dimensional space of operators  $h(n)$  for each  $n$  is also carried out.) In vertex operator algebra theory and conformal field theory it is standard procedure to embed operators such as  $h(n)$  and  $L(n)$  into generating functions and to compute with these generating functions, using a formal calculus.

The space  $S$  is naturally  $\mathbb{Z}$ -graded, with  $\deg h(j) = j$  for  $j < 0$ , and  $S$  is graded as a  $\mathfrak{v}$ -module. It is appropriate to use the negative of this grading, that is, to define a new grading (by “conformal weights”) on the space  $S$  by the rule  $\text{wt } h(-j) = j$  for  $j > 0$ ; for each  $n \geq 0$ , the homogeneous subspace of  $S$  of weight  $n$  coincides with the eigenspace of the operator  $L(0)$  with eigenvalue  $n$ . For  $n \in \mathbb{Z}$  (or  $n \geq 0$ ) let  $S_n$  be the homogeneous subspace of  $S$  of weight  $n$ , and consider the formal power series

$$\dim_* S = \sum_{n \geq 0} (\dim S_n) q^n \quad (1.14)$$

(the “graded dimension” of the graded space  $S$ ). Clearly,

$$\dim_* S = \prod_{n > 0} (1 - q^n)^{-1}. \quad (1.15)$$

As is well known, removing the normal ordering in the definition of  $L(0)$  introduces an infinity which formally equals  $\frac{1}{2}\zeta(-1)$ , since the unrigorous expression

$$\bar{L}(0) = \frac{1}{2} \sum_{j \in \mathbb{Z}} h(-j)h(j) \quad (1.16)$$

formally equals (by (1.8))

$$L(0) + \frac{1}{2}(1 + 2 + 3 + \cdots), \quad (1.17)$$

which itself formally equals

$$L(0) + \frac{1}{2}\zeta(-1) = L(0) - \frac{1}{24}. \quad (1.18)$$

Rigorizing  $\bar{L}(0)$  by defining it as

$$\bar{L}(0) = L(0) + \frac{1}{2}\zeta(-1), \quad (1.19)$$

we set

$$\bar{L}(n) = L(n) \quad \text{for } n \neq 0, \quad (1.20)$$

to get a new basis of  $\mathfrak{v}$ . (We are identifying the elements of  $\mathfrak{v}$  with operators on  $S$ .) The brackets become:

$$[\bar{L}(m), \bar{L}(n)] = (m - n)\bar{L}(m + n) + \frac{1}{12}m^3\delta_{m+n,0}; \quad (1.21)$$

that is,  $m^3 - m$  in (1.7) has become the pure monomial  $m^3$ .

It is a fundamental and well-known fact that this formal removal of the normal ordering gives rise to modular transformation properties: We define a new grading on  $S$  by using the eigenvalues of  $\bar{L}(0)$  in place of  $L(0)$ , so that the grading of  $S$  is “shifted” from the previous grading by conformal weights by the subtraction of  $\frac{1}{24}$  from the weights. Letting  $\chi(S)$  be the corresponding graded dimension, we have

$$\chi(S) = \frac{1}{\eta(q)}, \quad (1.22)$$

where

$$\eta(q) = q^{\frac{1}{24}} \prod_{n>0} (1 - q^n). \quad (1.23)$$

With the formal variable  $q$  replaced by  $e^{2\pi i\tau}$ ,  $\tau$  in the upper half-plane,  $\eta(q)$ , Dedekind's eta-function, has important (classical) modular transformation properties, unlike  $\prod_{n>0}(1 - q^n)$ .

In [Bl], Bloch considered the larger Lie algebra of formal differential operators, spanned by

$$\{t^n D^m | n \in \mathbb{Z}, m \geq 0\} \quad (1.24)$$

or more precisely, we restrict to  $m > 0$  and further, to the Lie subalgebra  $\mathcal{D}^+$ , containing  $\mathfrak{d}$ , spanned by the differential operators of the form  $D^r(t^n D)D^r$  for  $r \geq 0$ ,  $n \in \mathbb{Z}$ . Then we can construct a central extension of  $\mathcal{D}^+$  using generalizations of the operators (1.13):

$$L^{(r)}(n) = \frac{1}{2} \sum_{j \in \mathbb{Z}} j^r h(j)(n-j)^r h(n-j) \quad \text{for } n \neq 0, \quad (1.25)$$

$$L^{(r)}(0) = \frac{1}{2} \sum_{j \in \mathbb{Z}} (-j)^r h(-|j|) j^r h(|j|), \quad (1.26)$$

that is,

$$L^{(r)}(n) = \frac{1}{2} \sum_{j \in \mathbb{Z}} j^r h(j)(n-j)^r h(n-j) : \quad (1.27)$$

for  $n \in \mathbb{Z}$ . These operators provide [Bl] a central extension of  $\mathcal{D}^+$  such that

$$L^{(r)}(n) \mapsto (-1)^{r+1} D^r(t^n D)D^r. \quad (1.28)$$

A central point of [Bl] is that the formal removal of the normal-ordering procedure in the definition (1.26) of  $L^{(r)}(0)$  adds the infinity  $(-1)^r \frac{1}{2} \zeta(-2r-1) = -\sum_{n>0} n^{2r+1}$  (generalizing (1.16)–(1.18)), and if we correspondingly define

$$\bar{L}^{(r)}(0) = L^{(r)}(0) + (-1)^r \frac{1}{2} \zeta(-2r-1) \quad (1.29)$$

and  $\bar{L}^{(r)}(n) = L^{(r)}(n)$  for  $n \neq 0$  (generalizing (1.19) and (1.20)), the commutators simplify in a remarkable way: The complicated polynomial in the scalar term of  $[\bar{L}^{(r)}(m), \bar{L}^{(s)}(-m)]$  reduces to a pure monomial in  $m$ , by analogy with, and generalizing, the passage from  $m^3 - m$  to  $m^3$  in (1.21); see [Bl] for the formulas and further results. (See also [PRS] and [FS] for treatments of essentially the same phenomena in physical contexts.)

In [L2] two layers of “explanation” and generalization of the results of [Bl] were presented. It was recalled that for  $k > 1$ ,

$$\zeta(-k+1) = -\frac{B_k}{k}, \quad (1.30)$$

where the  $B_k$  are the Bernoulli numbers, defined by the generating function

$$\frac{x}{e^x - 1} = \sum_{k \geq 0} \frac{B_k}{k!} x^k, \quad (1.31)$$

where  $x$  is a formal variable, and (a variant of) Euler’s heuristic interpretation of (1.30) in terms of the divergent series (1.1) was also recalled. The “first layer of explanation” proceeded as follows:

Using a formal variable  $x$ , we form the generating functions

$$h(x) = \sum_{n \in \mathbb{Z}} h(n) x^{-n} \quad (1.32)$$

and

$$L^{(r)}(x) = \sum_{n \in \mathbb{Z}} L^{(r)}(n) x^{-n}, \quad (1.33)$$

and using  $D_x$  to denote the operator  $x \frac{d}{dx}$  (as in (1.5)), we observe that

$$L^{(r)}(x) = \frac{1}{2} \colon (D_x^r h(x))^2 \colon, \quad (1.34)$$

where the colons, as always, denote normal ordering. (For other purposes, other versions of these generating functions are used, in particular,  $h(x) = \sum_{n \in \mathbb{Z}} h(n) x^{-n-1}$  in place of (1.32); see (2.22) below.)

Next we introduce suitable generating functions over the number of *derivatives*, and we use the formal multiplicative analogue

$$e^{y D_x} f(x) = f(e^y x) \quad (1.35)$$

of the formal Taylor theorem

$$e^{y \frac{d}{dx}} f(x) = f(x + y), \quad (1.36)$$

where  $f(x)$  is an arbitrary formal series of the form  $\sum_n a_n x^n$ ,  $n$  is allowed to range over something very general, like  $\mathbb{Z}$  or even  $\mathbb{C}$ , say, and the  $a_n$  lie in

a fixed vector space (cf. [FLM], Proposition 8.3.1). Although  $\colon(D_x^r h(x))^2\colon$  (recall (1.34)) is hard to put into a “good” generating function over  $r$ , we make the problem easier by making it more general: Consider independently many derivatives on each of the two factors  $h(x)$  in  $\colon h(x)^2\colon$ , use two new independent formal variables  $y_1$  and  $y_2$ , and form the generating function

$$L^{(y_1, y_2)}(x) = \frac{1}{2} \colon(e^{y_1 D_x} h(x))(e^{y_2 D_x} h(x))\colon = \frac{1}{2} \colon h(e^{y_1} x) h(e^{y_2} x) \colon \quad (1.37)$$

(where we use (1.35)), so that  $L^{(r)}(x)$  is a “diagonal piece” of this generating function. Using formal vertex operator calculus techniques, we can calculate

$$[\colon h(e^{y_1} x_1) h(e^{y_2} x_1) \colon, \colon h(e^{y_3} x_2) h(e^{y_4} x_2) \colon]. \quad (1.38)$$

Now the formal expression  $h(e^{y_1} x) h(e^{y_2} x)$  is not rigorous, as we see by (for example) trying to compute the constant term in the variables  $y_1$  and  $y_2$  in this expression; the failure of this expression to be defined in fact corresponds exactly to the occurrence of formal sums like  $\sum_{n>0} n^r$  with  $r > 0$ , as we have been discussing. However, we have

$$h(x_1) h(x_2) = \colon h(x_1) h(x_2) \colon + x_2 \frac{\partial}{\partial x_2} \frac{1}{1 - x_2/x_1} \quad (1.39)$$

and it follows that

$$h(e^{y_1} x_1) h(e^{y_2} x_2) = \colon h(e^{y_1} x_1) h(e^{y_2} x_2) \colon + x_2 \frac{\partial}{\partial x_2} \frac{1}{1 - e^{y_2} x_2 / e^{y_1} x_1}; \quad (1.40)$$

note that  $x_2 \frac{\partial}{\partial x_2}$  can be replaced by  $-\frac{\partial}{\partial y_1}$  in the last expression. The expression  $\frac{1}{1 - e^{y_2} x_2 / e^{y_1} x_1}$  came from, and is, a geometric series expansion (recall (1.39)).

If we try to set  $x_1 = x_2 (= x)$  in (1.40), the result is unrigorous on the left-hand side, as we have pointed out, *but the result has rigorous meaning on the right-hand side*, because the normal-ordered product  $\colon h(e^{y_1} x) h(e^{y_2} x) \colon$  is certainly well defined, and the expression  $-\frac{\partial}{\partial y_1} \frac{1}{1 - e^{-y_1 + y_2}}$  can be interpreted rigorously as in (1.31); more precisely, we take  $\frac{1}{1 - e^{-y_1 + y_2}}$  to mean the formal (Laurent) series in  $y_1$  and  $y_2$  of the shape

$$\frac{1}{1 - e^{-y_1 + y_2}} = (y_1 - y_2)^{-1} F(y_1, y_2), \quad (1.41)$$

where  $(y_1 - y_2)^{-1}$  is understood as the binomial expansion (geometric series) in nonnegative powers of  $y_2$  and  $F(y_1, y_2)$  is an (obvious) formal power series in (nonnegative powers of)  $y_1$  and  $y_2$ . This motivates us to define a new “normal-ordering” procedure

$${}_+^+ h(e^{y_1} x) h(e^{y_2} x) {}_+^+ = :h(e^{y_1} x) h(e^{y_2} x): - \frac{\partial}{\partial y_1} \frac{1}{1 - e^{-y_1 + y_2}}, \quad (1.42)$$

with the last part of the right-hand side being understood as we just indicated. *This “rigorization” of the undefined formal expression  $h(e^{y_1} x) h(e^{y_2} x)$  corresponds exactly to Euler’s heuristic interpretation of (1.30) discussed in [L2].* This gives us a natural “explanation” of the zeta-function-modified operators defined in (1.29): We use (1.42) to define the following analogues of the operators (1.37):

$$\bar{L}^{(y_1, y_2)}(x) = \frac{1}{2} {}_+^+ h(e^{y_1} x) h(e^{y_2} x) {}_+^+, \quad (1.43)$$

and the operator  $\bar{L}^{(r)}(n)$  is exactly  $(r!)^2$  times the coefficient of  $y_1^r y_2^r x_0^{-n}$  in (1.43); the significant case is the case  $n = 0$ .

With the new normal ordering (1.42) replacing the old one, remarkable cancellation occurs in the commutator (1.38), and here is the main result of the “first layer of explanation” of Bloch’s formula for  $[\bar{L}^{(r)}(m), \bar{L}^{(s)}(n)]$  in [Bl], in a somewhat generalized form:

**Theorem 1.1** [L2] *With the formal delta-function Laurent series  $\delta(x)$  defined as*

$$\delta(x) = \sum_{n \in \mathbb{Z}} x^n, \quad (1.44)$$

*and with independent commuting formal variables as indicated, we have:*

$$\begin{aligned} & [\bar{L}^{(y_1, y_2)}(x_1), \bar{L}^{(y_3, y_4)}(x_2)] \\ &= -\frac{1}{2} \frac{\partial}{\partial y_1} \left( \bar{L}^{(-y_1 + y_2 + y_3, y_4)}(x_2) \delta\left(\frac{e^{y_1} x_1}{e^{y_3} x_2}\right) \right. \\ & \quad \left. + \bar{L}^{(-y_1 + y_2 + y_4, y_3)}(x_2) \delta\left(\frac{e^{y_1} x_1}{e^{y_4} x_2}\right) \right) \\ & \quad - \frac{1}{2} \frac{\partial}{\partial y_2} \left( \bar{L}^{(y_1 - y_2 + y_3, y_4)}(x_2) \delta\left(\frac{e^{y_2} x_1}{e^{y_3} x_2}\right) \right. \\ & \quad \left. + \bar{L}^{(y_1 - y_2 + y_4, y_3)}(x_2) \delta\left(\frac{e^{y_2} x_1}{e^{y_4} x_2}\right) \right). \end{aligned} \quad (1.45)$$

As discussed in [L2], the pure monomials in  $m$  that we set out to explain now emerge completely naturally, since for example the delta-function expression  $\delta(e^{y_1}x_1/e^{y_3}x_2)$  can be written as  $e^{y_1D_{x_1}}e^{y_3D_{x_2}}\delta(x_1/x_2)$ , and when we extract and equate the coefficients of the monomials in the variables  $y_1^r y_2^r y_3^s y_4^s$  on the two sides of (1.45), we get expressions like  $(D^j\delta)(x_1/x_2)$ , whose expansion, in turn, in powers of  $x_1$  and  $x_2$  clearly yields a pure monomial analogous to and generalizing the expression  $m^3$  in (1.21). (All of these considerations hold equally well in the more general situation where we start with a Heisenberg algebra based on a finite-dimensional space rather than a one-dimensional space.)

In the next section we recall from [L2] how these considerations can actually be understood in the much greater generality of vertex operator algebra theory in general (the “second explanation and generalization”).

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## 2 The general principles

At this point we recall the definition of the notion of vertex operator algebra from [FLM], [FHL]. This is a variant of Borchers’ notion of vertex algebra [Bo] and is based on the “Jacobi identity” as formulated in [FLM] and [FHL]; we need this formal-variable formulation in order to express our results in a natural way. We continue to use commuting formal variables  $x, x_0, x_1, x_2$  and several other variables. Recall the formal delta-function Laurent series  $\delta(x)$  (1.44).

**Definition 2.1** A *vertex operator algebra*  $(V, Y, \mathbf{1}, \omega)$ , or simply  $V$  (over  $\mathbb{C}$ ), is a  $\mathbb{Z}$ -graded vector space (graded by *weights*)

$$V = \coprod_{n \in \mathbb{Z}} V_{(n)}; \text{ for } v \in V_{(n)}, \ n = \text{wt } v; \quad (2.1)$$

such that

$$\dim V_{(n)} < \infty \text{ for } n \in \mathbb{Z}, \quad (2.2)$$

$$V_{(n)} = 0 \text{ for } n \text{ sufficiently small}, \quad (2.3)$$

equipped with a linear map  $V \otimes V \rightarrow V[[x, x^{-1}]]$ , or equivalently,

$$\begin{aligned} V &\rightarrow (\text{End } V)[[x, x^{-1}]] \\ v &\mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} \quad (\text{where } v_n \in \text{End } V), \end{aligned} \quad (2.4)$$

$Y(v, x)$  denoting the *vertex operator associated with*  $v$ , and equipped also with two distinguished homogeneous vectors  $\mathbf{1} \in V_{(0)}$  (the *vacuum*) and  $\omega \in V_{(2)}$ . The following conditions are assumed for  $u, v \in V$ : the *lower truncation condition* holds:

$$u_n v = 0 \quad \text{for } n \text{ sufficiently large} \quad (2.5)$$

(or equivalently,  $Y(u, x)v$  involves only finitely many negative powers of  $x$ );

$$Y(\mathbf{1}, x) = \mathbf{1} \quad (\mathbf{1} \text{ on the right being the identity operator}); \quad (2.6)$$

the *creation property* holds:

$$Y(v, x)\mathbf{1} \in V[[x]] \quad \text{and} \quad \lim_{x \rightarrow 0} Y(v, x)\mathbf{1} = v \quad (2.7)$$

(that is,  $Y(v, x)\mathbf{1}$  involves only nonnegative integral powers of  $x$  and the constant term is  $v$ ); with binomial expressions understood to be expanded in nonnegative powers of the second variable, the *Jacobi identity* (the main axiom) holds:

$$\begin{aligned} x_0^{-1} \delta \left( \frac{x_1 - x_2}{x_0} \right) Y(u, x_1) Y(v, x_2) - x_0^{-1} \delta \left( \frac{x_2 - x_1}{-x_0} \right) Y(v, x_2) Y(u, x_1) \\ = x_2^{-1} \delta \left( \frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0)v, x_2) \end{aligned} \quad (2.8)$$

(note that when each expression in (2.8) is applied to any element of  $V$ , the coefficient of each monomial in the formal variables is a finite sum; on the right-hand side, the notation  $Y(\cdot, x_2)$  is understood to be extended in the obvious way to  $V[[x_0, x_0^{-1}]]$ ); the Virasoro algebra relations hold (acting on  $V$ ):

$$[L(m), L(n)] = (m - n)L(m + n) + \frac{1}{12}(m^3 - m)\delta_{n+m,0}(\text{rank } V)\mathbf{1} \quad (2.9)$$

for  $m, n \in \mathbb{Z}$ , where

$$L(n) = \omega_{n+1} \quad \text{for } n \in \mathbb{Z}, \quad \text{i.e.,} \quad Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2} \quad (2.10)$$

and

$$\text{rank } V \in \mathbb{C}; \quad (2.11)$$

$$L(0)v = nv = (\text{wt } v)v \text{ for } n \in \mathbb{Z} \text{ and } v \in V_{(n)}; \quad (2.12)$$

$$\frac{d}{dx}Y(v, x) = Y(L(-1)v, x) \quad (2.13)$$

(the  $L(-1)$ -derivative property).

In the presence of simpler axioms, the Jacobi identity in the definition is equivalent to a suitably-formulated “commutativity” relation and a suitably formulated “associativity relation” (recall [FLM], [FHL]). The commutativity condition asserts that for  $u, v \in V$ ,

$$Y(u, x_1)Y(v, x_2) \sim Y(v, x_2)Y(u, x_1), \quad (2.14)$$

where “ $\sim$ ” denotes equality up to a suitable kind of generalized analytic continuation, and the associativity condition asserts that

$$Y(u, x_1)Y(v, x_2) \sim Y(Y(u, x_1 - x_2)v, x_2), \quad (2.15)$$

where the right-hand side and the generalized analytic continuation have to be understood in suitable ways (see [FLM] and [FHL] and cf. [BPZ] and [G]); the right-hand side of (2.15) is *not* a well-defined formal series in  $x_1$  and  $x_2$ .

In [L2] we reformulated the associativity relation in the following way: Formally replacing  $x_1$  by  $e^y x_2$  in (2.15), we find (formally and unrigorously) that

$$Y(u, e^y x_2)Y(v, x_2) \sim Y(Y(u, (e^y - 1)x_2)v, x_2). \quad (2.16)$$

While the left-hand side of (2.16) is not a well-defined formal series in the formal variables  $y$  and  $x_2$ , the right-hand side of (2.16) *is* in fact a well-defined formal series in these formal variables. By replacing  $x_1$  by  $e^y x_2$  we have made the right-hand side of (2.16) rigorous (and the left-hand side unrigorous). Next, instead of the vertex operators  $Y(v, x)$ , we want the modified vertex operators defined for homogeneous elements  $v \in V$  by:

$$X(v, x) = x^{\text{wt } v} Y(v, x) = Y(x^{L(0)}v, x), \quad (2.17)$$

as in [FLM], formula (8.5.27) (recall that  $L(0)$ -eigenvalues define the grading of  $V$ ); the formula  $X(v, x) = Y(x^{L(0)}v, x)$  works for *all*  $v \in V$  (not necessarily homogeneous).

Using this we formally obtain from (2.16) the formal relation

$$X(u, e^y x_2)X(v, x_2) \sim X(Y[u, y]v, x_2), \quad (2.18)$$

where  $Y[u, y]$  is the operator defined in [Z1], [Z2] as follows:

$$Y[u, y] = Y(e^{yL(0)}u, e^y - 1); \quad (2.19)$$

the right-hand side of (2.18) is still well defined (and the left-hand side still not well defined). By Zhu's change-of-variables theorem in [Z1], [Z2] (see [L1] and [H1], [H2] for different treatments of this theorem),  $x \mapsto Y[u, x]$  defines a new vertex operator algebra structure on the same vector space  $V$  under suitable conditions; in [L1], a simple formal-variable proof of the Jacobi identity, which we shall need below, for these operators is given.

The formal relation (2.18) generalizes to products of several operators, as follows:

$$\begin{aligned} & X(v_1, e^{y_1}x)X(v_2, e^{y_2}x) \cdots X(v_n, e^{y_n}x)X(v_{n+1}, x) \\ & \sim X(Y[v_1, y_1]Y[v_2, y_2] \cdots Y[v_n, y_n]v_{n+1}, x). \end{aligned} \quad (2.20)$$

We can also multiply the variable  $x$  in this relation by an exponential, and we get such formal relations as:

$$X(u, e^{y_1}x)X(v, e^{y_2}x) \sim X(Y[u, y_1 - y_2]v, e^{y_2}x), \quad (2.21)$$

which should be compared with (2.15).

As discussed in [L2], the point is that very special cases of the formal relation (2.18) precisely “explain” the classical formal relation (1.1) in the setting of vertex operator algebra theory. Specifically, the polynomial algebra  $S$  (recall (1.9)) carries a canonical vertex operator algebra structure of rank 1 with vacuum vector  $\mathbf{1}$  equal to  $1 \in S$ , with the operators  $L(n)$  agreeing with the operators defined in (1.11), (1.12), with

$$Y(h(-1), x) = x^{-1}h(x) = \sum_{n \in \mathbb{Z}} h(n)x^{-n-1} \quad (2.22)$$

(recall (1.32)) and with  $\omega = \frac{1}{2}(h(-1))^2 \in S$ . Setting

$$v_0 = h(-1) \in S, \quad (2.23)$$

we thus have

$$X(v_0, x) = X(h(-1), x) = h(x). \quad (2.24)$$

Next, using this we state a precise formula that equates the new normal-ordering procedure (1.42) (which in turn interpreted Bloch's zeta-function-modified operators and the classical formal relation (1.1)) with its current, still more conceptual, formulation:

$${}^+_+ h(e^y x) h(e^w x) {}^+_+ = X(Y[v_0, y - w]v_0, e^w x), \quad (2.25)$$

or equivalently,

$${}^+_+ h(e^{y+w} x) h(e^w x) {}^+_+ = X(Y[v_0, y]v_0, e^w x). \quad (2.26)$$

*Moreover, the very general formal relation (2.21), applied to the particular parameters in (2.25), amounts precisely to the formal relation*

$$h(e^y x) h(e^w x) \sim {}^+_+ h(e^y x) h(e^w x) {}^+_+, \quad (2.27)$$

*which was in turn the rigorization of the undefined formal expression  $h(e^y x) h(e^w x)$  by means of the generating function of the Bernoulli numbers (recall (1.42)).*

Thus we have interpreted (1.1) and (1.42) by means of a very special case of a very general picture. We still need to “explain” the bracket relation (1.45) from this point of view. (As we have mentioned, everything works for a Heisenberg Lie algebra based, more generally, on a finite-dimensional space.)

To do this, we now see that we need to compute in a conceptual way the commutators of certain expressions of the type  $X(Y[u, y]v, x)$ . But the most natural thing to do (just as was the case in the analogous contexts in [FLM], for instance) is to seek a stronger “Jacobi-type identity,” analogous to (2.8), and to derive the desired commutators from it. The main result announced in [L2] (Theorem 4.2) was in fact just such an identity, for operators of the simpler type  $X(v, x)$  rather than of the type  $X(Y[u, y]v, x)$ :

**Theorem 2.2** *In any vertex operator algebra  $V$ , for  $u, v \in V$  we have:*

$$\begin{aligned} x_0^{-1} \delta \left( e^{y_{21}} \frac{x_1}{x_0} \right) X(u, x_1) X(v, x_2) - x_0^{-1} \delta \left( -e^{y_{12}} \frac{x_2}{x_0} \right) X(v, x_2) X(u, x_1) \\ = x_2^{-1} \delta \left( e^{-y_{01}} \frac{x_1}{x_2} \right) X(Y[u, y_{01}]v, x_2), \end{aligned} \quad (2.28)$$

where

$$y_{21} = \log \left( 1 - \frac{x_2}{x_1} \right), \quad y_{12} = \log \left( 1 - \frac{x_1}{x_2} \right), \quad y_{01} = -\log \left( 1 - \frac{x_0}{x_1} \right), \quad (2.29)$$

log denoting the logarithmic formal series.

The proof is not difficult, starting from the Jacobi identity (2.8). What was most interesting was that an identity such as this exists at all, with its strong parallels with the Jacobi identity (2.8) itself. The three formal variables defined by the logarithmic formal series (2.29) can to a certain extent be viewed as independent formal variables in their own right; along with symmetry considerations, this is why we choose to write the delta-function expressions in (2.8) in this way.

Formula (2.28) can be viewed as solving a problem implicit in formula (8.8.43) (Corollary 8.8.19) of [FLM]: The left-hand side of that formula involved the expansion coefficients with respect to  $x_0$ ,  $x_1$  and  $x_2$  of the left-hand side of (2.28), but we had been unable to put the right-hand side of formula (8.8.43) of [FLM] into an elegant generating-function form. Theorem 2.2 solves this problem, and in the process, in fact relates the left-hand side to zeta-function values, as explained in the present work.

In [L2] we emphasized the philosophy to “always use generating functions,” and one can view the conversion of the right-hand side of formula (8.8.43) of [FLM] into the illuminating generating-function form (2.28) as a nontrivial example of this philosophy. (A much more elementary use of generating functions, central to the present work, is formula (1.35), relating a generating function ranging over the number of derivatives of a formal series with a formal change of variables, and expressing the fact that  $D_x$  is a formal infinitesimal dilation.)

After this review and elaboration of the results in [L2], below we present our results: the extension of (2.28) to expressions of the type  $X(Y[u, y]v, x)$ , the use of the resulting identity to compute commutators of such expressions,

and the application of these general principles to the recovery of Theorem 3.1 of [L2] (see Theorem 1.1 above)—a commutator formula for certain vertex operators that in turn yields a central extension of a Lie algebra of differential operators and the phenomena found by Bloch in [Bl].

### 3 Main results

We want to bracket expressions of the type  $X(Y[u, y]v, x)$  and thereby to recover Theorem 1.1, for the reasons that we have been discussing. First, in order to extract the commutator  $[X(u, x_1), X(v, x_2)]$  from (2.28), we perform the usual procedure (recall [FLM], [FHL]) of extracting the coefficient of  $x_0^{-1}$  (the formal residue  $\text{Res}_{x_0}$  with respect to the variable  $x_0$ ) on both sides, leaving us with the desired commutator on the left-hand side. To compute this formal residue on the right-hand side, we use the following variant of a standard formal change-of-variables formula: Let  $A$  be a commutative associative algebra and let  $F(x)$  be a formal power series in  $A[[x]]$  with no constant term and such that the coefficient of  $x^1$  is invertible in  $A$ . Then for  $h(x) \in A((x))$ ,

$$\text{Res}_x h(x) = \text{Res}_y (h(F(y))F'(y)). \quad (3.1)$$

This formula holds more generally for  $h(x) \in V((x))$ , where  $V$  is any  $A$ -module. Using this for  $A$  the commutative associative algebra  $\mathbb{C}[x_1, x_1^{-1}]$ ,  $V$  the  $A$ -module of operator-valued formal Laurent series in  $x_1$  and  $x_2$ ,  $F(y) = x_1(1 - e^y)$ , and  $h(x_0)$  equal to the left-hand side of (2.28), an operator-valued formal Laurent series in  $x_0, x_1$  and  $x_2$  with only finitely many negative powers of  $x_0$ , we obtain:

**Theorem 3.1** *In the setting of Theorem 2.2,*

$$[X(u, x_1), X(v, x_2)] = \text{Res}_y \delta \left( e^{-y} \frac{x_1}{x_2} \right) X(Y[u, y]v, x_2). \quad (3.2)$$

Here the dummy variable  $y$  is actually the variable denoted  $y_{01}$  in Theorem 2.2.

We can generalize Theorem 2.2 by taking the vectors  $u$  and  $v$  in that result to be appropriate composite expressions, and by suitably using (1.35), to obtain:

**Theorem 3.2** *In any vertex operator algebra  $V$ , for any  $u_1, v_1, u_2, v_2 \in V$  we have:*

$$\begin{aligned} & x_0^{-1} \delta \left( e^{w_{21}} \frac{e^{w_1} x_1}{x_0} \right) X(Y[u_1, y_1]v_1, e^{w_1} x_1) X(Y[u_2, y_2]v_2, e^{w_2} x_2) \\ & - x_0^{-1} \delta \left( -e^{w_{12}} \frac{e^{w_2} x_2}{x_0} \right) X(Y[u_2, y_2]v_2, e^{w_2} x_2) X(Y[u_1, y_1]v_1, e^{w_1} x_1) \\ & = (e^{w_2} x_2)^{-1} \delta \left( e^{-w_{01}} \frac{e^{w_1} x_1}{e^{w_2} x_2} \right) X(Y[Y[u_1, y_1]v_1, w_{01}]Y[u_2, y_2]v_2, e^{w_2} x_2), \end{aligned} \quad (3.3)$$

where

$$w_{21} = \log \left( 1 - \frac{e^{w_2} x_2}{e^{w_1} x_1} \right), \quad w_{12} = \log \left( 1 - \frac{e^{w_1} x_1}{e^{w_2} x_2} \right), \quad w_{01} = -\log \left( 1 - \frac{x_0}{e^{w_1} x_1} \right). \quad (3.4)$$

As in Theorem 2.2, we choose to write the delta-function expressions in this way to exhibit the variables appropriate to the context.

From either Theorem 3.1 or Theorem 3.2 we obtain the corresponding commutator formula:

**Theorem 3.3** *In the setting of Theorem 3.2,*

$$\begin{aligned} & [X(Y[u_1, y_1]v_1, e^{w_1} x_1), X(Y[u_2, y_2]v_2, e^{w_2} x_2)] \\ & = \text{Res}_w \left( e^{-w} \frac{e^{w_1} x_1}{e^{w_2} x_2} \right) X(Y[Y[u_1, y_1]v_1, w]Y[u_2, y_2]v_2, e^{w_2} x_2). \end{aligned} \quad (3.5)$$

The dummy variable  $w$  is the same as the variable denoted  $w_{01}$  in Theorem 3.2. Of course, both Theorem 3.2 and Theorem 3.3 extend to multiple expressions of the type in the right-hand side of (2.20).

These and related general results can be applied to a wide variety of special situations. Here we indicate the main steps in the conceptual recovery of Bloch's formulas, as reformulated and generalized in Theorem 3.1 of [L2] (recalled in Theorem 1.1 above).

We of course work in the special setting discussed in (2.22)–(2.27) above. We can use Theorem 3.3 to compute

$$[X(Y[v_0, y_1]v_0, e^{w_1} x_1), X(Y[v_0, y_2]v_0, e^{w_2} x_2)] \quad (3.6)$$

(the case  $u_1 = v_1 = u_2 = v_2 = v_0$  in (3.5)). It is convenient to take  $w_1 = w_2 = 0$  at first, and then to apply formula (1.35) to restore  $e^{w_1}$  and  $e^{w_2}$  at

the end. In order to evaluate and simplify the right-hand side of (3.5), we systematically use Zhu's theorem (mentioned above) that the Jacobi identity holds for the operators  $Y[u, x]$ . The analysis of the right-hand side of (3.5) is straightforward and natural; the details are given in [L3]. Here we state a result, in the full generality of Theorem 3.3, that serves to partially evaluate the expression (3.5); without loss of generality, we state it for the case  $w_1 = w_2 = 0$ :

**Theorem 3.4** *In the indicated setting, we have*

$$\begin{aligned}
& [X(Y[u_1, y_1]v_1, x_1), X(Y[u_2, y_2]v_2, x_2)] \\
&= \text{Res}_{t_1} e^{t_1 \frac{\partial}{\partial y_1}} \left( \delta \left( e^{-t_1} \frac{x_1}{x_2} \right) X(Y[u_2, y_2]Y[u_1, y_1]Y[v_1, t_1]v_2, x_2) \right) \\
&+ \text{Res}_{t_2} e^{-t_2 \frac{\partial}{\partial y_1}} \left( \delta \left( e^{y_1} \frac{x_1}{x_2} \right) X(Y[u_2, y_2]Y[v_1, -y_1]Y[u_1, t_2]v_2, x_2) \right) \\
&- \text{Res}_{t_3} e^{-t_3 \frac{\partial}{\partial y_2}} \left( \delta \left( e^{-y_2} \frac{x_1}{x_2} \right) X(Y[Y[u_1, y_1]Y[u_2, t_3]v_1, y_2]v_2, x_2) \right) \\
&- \text{Res}_{t_4} e^{-t_4 \frac{\partial}{\partial y_2}} \left( \delta \left( e^{-y_2+y_1} \frac{x_1}{x_2} \right) X(Y[Y[Y[u_2, t_4]u_1, y_1]v_1, y_2 - y_1]v_2, x_2) \right).
\end{aligned}$$

Even in this full generality, the features of Theorem 1.1 are already starting to appear here. We can apply this result to the special case (3.6). As in the “direct” proof of Theorem 3.1 of [L2] (using the ingredients of Section 3 of [L2] rather than the present general considerations), rather subtle cancellation occurs on the right-hand side, but here the cancellation occurs in a more conceptual context. The result is:

**Theorem 3.5** *Applied to the case (3.6), Theorem 3.3 yields precisely the assertion of Theorem 3.1 of [L2].*

We have thus “explained” both the zeta-function-modified operators and their commutators as studied in [Bl].

## References

- [BPZ] A. A. Belavin, A. M. Polyakov and A. B. Zamolodchikov, Infinite conformal symmetries in two-dimensional quantum field theory, *Nucl. Phys.* **B241** (1984), 333–380.

- [Bl] S. Bloch, Zeta values and differential operators on the circle, *J. Algebra* **182** (1996), 476–500.
- [Bo] R. E. Borcherds, Vertex algebras, Kac-Moody algebras, and the Monster, *Proc. Natl. Acad. Sci. USA* **83** (1986), 3068–3071.
- [FS] P. Fendley and H. Saleur, Massless integrable quantum field theories and massless scattering in 1+1 dimensions, *Proc. Conference Strings '93*, ed. M. B. Halpern, G. Rivlis and A. Sevrin, World Scientific, Singapore, 1995, 87 (hep-th/9310058).
- [FHL] I. B. Frenkel, Y.-Z. Huang and J. Lepowsky, On axiomatic approaches to vertex operator algebras and modules, preprint, 1989; *Memoirs Amer. Math. Soc.* **104**, 1993.
- [FLM] I. B. Frenkel, J. Lepowsky and A. Meurman, *Vertex Operator Algebras and the Monster*, Pure and Appl. Math., Vol. 134, Academic Press, Boston, 1988.
- [G] P. Goddard, Meromorphic conformal field theory, *Infinite Dimensional Lie Algebras and Groups, Advanced Series in Math. Physics*, Vol. 7, ed. V. Kac, World Scientific, Singapore, 1989, 556–587.
- [H1] Y.-Z. Huang, Applications of the geometric interpretation of vertex operator algebras, *Proc. 20th International Conference on Differential Geometric Methods in Theoretical Physics, New York, 1991*, ed. S. Catto and A. Rocha, World Scientific, Singapore, 1992, 333–343.
- [H2] Y.-Z. Huang, *Two-dimensional Conformal Geometry and Vertex Operator Algebras*, Progress in Math., Vol. 148, Birkhäuser, Boston, 1997.
- [L1] J. Lepowsky, Remarks on vertex operator algebras and moonshine, *Proc. 20th International Conference on Differential Geometric Methods in Theoretical Physics, New York, 1991*, ed. S. Catto and A. Rocha, World Scientific, Singapore, 1992, 362–370.
- [L2] J. Lepowsky, Vertex operator algebras and the zeta function, *Recent Developments in Quantum Affine Algebras and Related Topics*, ed.

N. Jing and K. C. Misra, Contemporary Math., Vol. 248, Amer. Math. Soc., 1999, 327-340.

- [L3] J. Lepowsky, A “Jacobi identity” for vertex operator algebras related to zeta-function values, to appear.
- [PRS] C. N. Pope, L. J. Romans and X. Shen, The complete structure of  $W_\infty$ , *Physics Letters B* **236** (1990), 173–177.
- [Z1] Y. Zhu, Vertex operators, elliptic functions and modular forms, Ph.D. thesis, Yale University, 1990.
- [Z2] Y. Zhu, Modular invariance of characters of vertex operator algebras, *J. Amer. Math. Soc.* **9** (1996), 237–307.

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